Zoltán Balogh Memorial Topology Conference
Contributed Problems

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Title: **A bunch of Problems on D-spaces**

We consider below only T_1-spaces. The notion of a D-space was introduced by E. van Douwen in [5]. A *neighbourhood assignment* on a topological space X is a mapping ϕ of X into the topology $\mathcal{T}$ of X such that $x \in \phi(x)$, for each $x \in X$. A space X is a *D-space* if, for every neighbourhood assignment ϕ on X, there exists a closed discrete subset A of X such that the family $\phi(A)$ covers X. A principal property of D-spaces is that the extent of any D-space coincides with the Lindelöf number. In particular, every countably compact D-space is compact, and every D-space with the countable extent is Lindelöf.

Problem (1)**.** (E. van Douwen [5]) *Is every regular Lindelöf space a D-space?*

It is even unknown if every hereditarily Lindelöf regular T_1-space is a D-space.

Problem (2)**.** (E. van Douwen [5]) *Is every regular (Tychonoff) subparacompact space a D-space?*

Recall that a space X is *subparacompact* if every open covering of X can be refined by a σ-discrete closed covering [3]. All metrizable spaces, and, more generally, all Moore spaces are D-spaces [2]. A much more general result was recently obtained by R.Z. Buzyakova: every strong Σ-space is a D-space [4]. Hence, all Tychonoff spaces with a countable network, all σ-spaces, all paracompact p-spaces, and all Lindelöf Σ-spaces are D-spaces. On the other hand, there exists a locally compact σ-metrizable Tychonoff space with the diagonal G_δ which is not a D-space [6].

A space X is an *aD-space* [1] if for each closed subset F of X and each open covering γ of X there exist a closed discrete subset A of F and a mapping ϕ of A into γ such that $a \in \phi(a)$, for each $a \in A$, and the family $\phi(A) = \{\phi(a) : a \in A\}$ covers F. Every subparacompact space is an aD-space [2]. Notice that every aD-space of the countable extent is Lindelöf.

Problem (3)**.** [1] *Is every aD-space X a D-space? What if X is, in addition, regular (Tychonoff)?*

The positive answer to Problem 3 would also solve Problems 1 and 2.

Several open problems on D-spaces are related to the following general question: how complex can be a space which is the union of two "nice" subspaces? The Alexandroff compactification of any uncountable discrete space ω_1 is the union of two metrizable (in fact, discrete) subspaces, while it is not first countable and, therefore, not metrizable. Another example of a non-metrizable compactum, which is the union of two metrizable subspaces, is the double circumference of Alexandroff and Urysohn. This space is first countable and, hence, Fréchet-Urysohn.

If a regular space X is the union of finitely many metrizable subspaces, then X is a D-space [1]. If a regular space X is the union of a countable family γ of dense metrizable subspaces, then X is a D-space [1]. Similarly, if a space X is the union of a countable family of open metrizable subspaces, then X is also a D-space [1].

Problem (4)**.** [1] *Suppose that $X = Y \cup Z$, where Y and Z are D-spaces. Is then X a D-space? Is then X an aD-space? What if X is, in addition, assumed to be Tychonoff?*

Problem (5)**.** [1] *Suppose that $X = Y \cup Z$, where Y and Z are D-spaces, and X is countably compact. Must X be compact? What if X is, in addition, assumed to be Tychonoff?*

We do not even know whether every countably compact space X which is the union of a countable family of D-spaces must be compact.

Problem (6). *Suppose that X is the union of a countable family of open D-subspaces. Must X be a D-space?*

In connection with Problems 4 and 5, notice the following simple result [1]: If $X = Y \cup Z$, where Y and Z are aD-spaces (D-spaces) and Y is closed in X, then X is also an aD-space (a D-space, respectively).

Problem (7). *Suppose that X is a Lindelöf D-space and Y is a continuous image of X. Must Y be a D-space? What if Y is, in addition, assumed to be regular (Tychonoff)?*

Here is an interesting special version of Problems 4 and 7.

Problem (8). *Suppose that $X = Y \cup Z$, where Y and Z are Lindelöf D-spaces. Is then X a D-space? What if X is, in addition, assumed to be regular?*

Below $C_p(\omega_1)$ is the space of continuous real-valued functions, in the pointwise topology, on the space ω_1 of countable ordinals (with the order topology).

Problem (9). *Is $C_p(\omega_1)$ a D-space?*

Note, that $C_p(\omega_1)$ is Lindelöf.

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Title: **The Namioka properties**

A completely regular space B has the Namioka property if for every separately continuous $f : B \times K \to [0,1]$, with K compact, f is jointly continuous at each point of a set $A \times K$, where A is dense in B.

Problem (A). *Let B be a completely regular space such that for every separately continuous $f : B \times K \to [0,1]$, with K compact, and each closed F is $B \times K$ projecting irreducibly onto B, the set of points of continuity of the restriction $f|F : F \to [0,1]$ is dense in F. Does B have the Namioka property?*

The Namioka property implies the property in Problem A, and this property yields that B is a Baire space, cf. [BP1]. Let us also notice that there is a Baire space B (even a Choquet space) and a separately continuous $f : B \times \beta B \to [0,1]$ such that the restriction of f to the diagonal $\{(b,b); b \in B\}$, has no continuity points [BP1].

Dually, a compact space K is a Namioka space, if for any separately continuous $f : B \times K \to [0,1]$, with B Baire, f is continuous at each point of some set $A \times K$, where A is dense in B. We shall denote by $(C(K), \mathrm{weak})$, $(C(K)^*, \mathrm{weak}^*)$ the space of real-valued continuous functions on K, and its dual space (identified with the space of Radon measures on K), endowed with the topology of pointwise convergence on $C(K)^*$, or $C(K)$, respectively.

Problem (B)**.** *Let K be a compact Namioka space. Is the duality mapping $(f,\mu) \to \int f\, d(\mu)$ Borel measurable on $(C(K), \mathrm{weak}) \times (C(K)^*, \mathrm{weak}^*)$?*

Let us notice that the duality mapping is not Borel for any infinite compact F-space K, cf. [BP2]. For relevant information on this topic we refer also to a paper by Maxim Burke, *B*orel measurability of separately continuous functions [Bu]. For additional information on the Namioka properties see the article by Mercourakis and Negrepontis [MN].

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Title: **Products of Baire spaces**

A space X is hereditarily Baire if all its closed subspaces are Baire. For metrizable X, this means that X contains no closed copy of the rationals.

Problem. *If X, Y are metrizable spaces, X is Baire, and Y is hereditarily Baire, is then the product $X \times Y$ a Baire space?*

There are metrizable Baire spaces X, Y whose product is not Baire, cf. Fleissner and Kunen [2]. On the other hand, arbitrary products of metrizable hereditarily Baire spaces are Baire, cf. [1]. Let us also notice that if a metrizable hereditarily Baire space is split into countably many sets, then one of them must contain a relatively open nonempty subspace whose product with any metrizable hereditarily Baire space is Baire, cf. [1].

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Title: **Some Cardinality Questions Concerning Countably Compact Spaces**

Dedicated to the Memory of Zoltan Balogh

Remark. The reader will lose little by interpreting the following Questions, and the results cited from the literature, within the class of Tychonoff spaces. In fact, however, several of those theorems are meaningful and valid when the expression "countably compact" is taken in the widest possible sense, that is, with no separation properties assumed; others must be construed within the class of Tychonoff spaces.

It was shown by Scarborough and A. H. Stone [8](5.6) that a product $\Pi_{i\in I} X_i$ of nonempty spaces is countably compact if and only if each subproduct $\Pi_{i\in J} X_i$ with $J \subseteq I$ and $|J| \leq 2^{2^{\mathfrak{c}}}$ is countably compact. Later, using the theory of p-limits and p-compactness ($p \in \omega^* := \beta(\omega)\backslash\omega$) introduced by Bernstein [1], Ginsburg and Saks [5](2.6) showed for a space X that every power X^κ is countably compact if and only if $X^{2^{\mathfrak{c}}}$ is countably compact. Prior to the appearance of [1], Frolík [4] had introduced a notion equivalent to the p-limit concept and had used it to show in ZFC that ω^* is not homogeneous, and also to produce a sequence of spaces X_n $(n < \omega)$ such that each $\Pi_{i\in F} X_i$ with $F \in [I]^{<\omega}$ is countably compact, while $\Pi_{n<\omega} X_n$ is not countably compact. With the results of [8], [5] and [4] before them, Saks [7] and Comfort [2] then proved:

Theorem 1. A product $\Pi_{i\in I} X_i$ of nonempty spaces is countably compact if and only if each subproduct $\Pi_{i\in J} X_i$ with $J \subseteq I$, $|J| \leq 2^{\mathfrak{c}}$, is countably compact.

Theorem 1 makes natural the following question, which was asked already in [2] and subsequently elsewhere.

Question (1). *In Theorem 1, is the cardinal number $2^{\mathfrak{c}}$ best possible?*

Question 1 has some close relatives, as follows.

Question (2). *Is there a set $\{X_i : i \in I\}$ of spaces, with $|I| = 2^{\mathfrak{c}}$, such that $\Pi_{i\in I} X_i$ is not countably compact but $\Pi_{i\in J} X_i$ is countably compact for each proper $J \subseteq I$?*

Question (3). *Is there a space X such that $X^{2^{\mathfrak{c}}}$ is not countably compact, but X^κ is countably compact for each $\kappa < 2^{\mathfrak{c}}$?*

Much is known about these questions. For example, Saks [7] answered Question 2 affirmatively, if MA is assumed. Yang [11] showed that Question 3 has an affirmative answer if and only if every set $S \subseteq \omega^*$ with $|S| < 2^{\mathfrak{c}}$ satisfies $\omega^* \neq A(S)$. Here $A(S)$ is the set of ultrafilters which sit "above" some element of S in the Rudin-Keisler order; that is, $A(S)$ is the set of $q \in \omega^*$ such that for some $f : \omega \to \omega$ the Stone extension $\overline{f} : \beta(\omega) \to \beta(\omega)$ satisfies $\overline{f}(q) \in S$. Accordingly, all three questions have consistent positive solutions. What is desirable in each case is an absolute proof (in ZFC), or a consistent proof of the negation.

The questions can be formulated in the context of topological groups. Here is a relevant ZFC result.

Theorem 2. In order that every product of countably compact topological groups be countably compact, it is necessary and sufficient that there exists $p \in \omega^*$ such that each countably compact topological group is p-compact.

[The proof of necessity follows the argument of the proof of the theorem cited above from [2] and [7], while sufficiency is a consequence of Bernstein's theorem [1] that (for fixed $p \in \omega^*$) the product of any set of p-compact spaces is p-compact (and hence countably compact).]

Assuming CH, van Douwen [3] showed the existence of two countably compact topological groups whose product is not countably compact. K. P. Hart and van Mill [6] then showed, using $\mathrm{MA}_{\mathrm{countable}}$, that there is a countably compact group G such that $G \times G$ is not countably compact. These results were improved by Tomita [9], [10], who showed *inter alia*, always assuming $\mathrm{MA}_{\mathrm{countable}}$, that for $0 < k < \omega$ there is a sequence G_n $(n < \omega)$ of countably compact topological groups such that each $\Pi_{n\in F} G_n$ is countably compact when $F \in [\omega]^{\leq k}$, but $\Pi_{n\in F} G_n$ is not countably compact when $F \in [\omega]^{\geq k+1}$. Those results suggest these questions.

Question (4). *Does there exist in* ZFC *a set G_i $(i \in I)$ of countably compact topological groups such that $\Pi_{i\in I} G_i$ is not countably compact? With $|I| < 2^{\mathfrak{c}}$? With $|I| = 2$? With $\Pi_{I\in J} G_i$ countably compact for all $J \subseteq I$ with $|J| < |I|$? With $\Pi_{i\in J} G_i$ countably compact for all proper $J \subseteq I$?*

Question (5). *Are the (equivalent) conditions given in Theorem 2 consistent with* ZFC*?*

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Title: **Countably compact groups without nontrivial convergent sequences**

Being a dyadic space, every infinite compact topological group has nontrivial convergent sequences. This inspired interest in compact-like group topologies without non-trivial convergent sequences (see, for example, [8, 5, 4, 6, 7, 9, 1, 3, 11, 2]).

Denote by $\mathcal{C}$ the class of Abelian groups that admit a countably compact group topology and by $\mathcal{P}$ the class of Abelian groups that admit a pseudocompact group topology. In [2] the following statement has been shown to be consistent with ZFC: If G is a group from the class $\mathcal{C}$ (or the class $\mathcal{P}$) and $|G| \leq 2^{\mathfrak{c}}$, then G admits a hereditarily separable countably compact (respectively, pseudocompact) group topology without non-trivial convergent sequences. This naturally brings us to the following

Question 1. Does every group G from the class $\mathcal{C}$ (or $\mathcal{P}$) has a countably compact (respectively, pseudocompact) group topology without non-trivial convergent sequences?

The next question, going in the opposite direction, may be considered as a "countably compact heir" of the fact that compact groups have non-trivial convergent sequences that still has a chance of a positive answer in ZFC.

Question 2. Let G be an infinite group from class $\mathcal{C}$. Does G has a countably compact group topology that contains a non-trivial convergent sequence?

Since all known examples of countably compact groups without non-trivial convergent sequences seem to be Abelian, it seems reasonable to ask what happens for *essentially* non-commutative groups.

Question 3. Let G be an infinite countably compact group without open Abelian subgroups. Does G have a non-trivial convergent sequence?

Question 4. Let G be an infinite countably compact group with trivial center $Z(G) = \{g \in G : xg = gx \text{ for each } x \in G\}$. Does G have a non-trivial convergent sequence?

Martin's Axiom MA yields positive answers to both Question 3 and Question 4 for groups of weight $< \mathfrak{c}$ [1]. A *consistent* negative answer to Question 4 was given in [2]. Furthermore, the example may be chosen hereditarily separable and locally connected (or hereditarily separable and totally disconnected).

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Title: **Compact-like group topologies on Abelian groups**

Halmos [11] asked which Abelian groups admit compact group topologies, and complete solution to this problem has been found by Harrison [12] and Hulanicki [13]. The counterpart of Halmos' problem for pseudocompact groups was attacked in [3, 4, 1, 2, 5] and complete solutions have been found for major classes of groups. Nevertheless, the general case still remains open:

Problem 5. Describe an algebraic structure of pseudocompact Abelian groups.

The question of which Abelian groups admit a countably compact group topology happens to be much more complicated. Let G be an Abelian group. As usual $r(G)$ denotes the *free rank* of G. For every natural number $n \geq 1$ define $G[n] = \{g \in G : ng = 0\}$ and $nG = \{ng : g \in G\}$. It is relatively easy to see that every countably compact group G satisfies the following two conditions [3, 5, 3]:

PS: Either $r(G) \geq \mathfrak{c}$ or $G = G[n]$ for some $n \in \omega \setminus \{0\}$.

CC: For every pair of integers $n \geq 1$ and $m \geq 1$ the group $mG[n]$ is either finite or has size at least $\mathfrak{c}$.

Definition 6. (i) We denote by $\mathcal{C}$ the class of Abelian groups that admit a countably compact group topology.

(ii) We denote by $\mathcal{P}$ the class of Abelian groups that admit a pseudocompact group topology.

By the above result, every group G from the class $\mathcal{C}$ satisfies both **PS** and **CC**.

After a series of scattered results [4, 8, 9, 18], a complete description of the algebraic structure of members of $\mathcal{C}$ that have size at most $\mathfrak{c}$ has been recently obtained, under Martin's Axiom MA, in [7]: MA implies that an Abelian group G of size at most $\mathfrak{c}$ belongs to $\mathcal{C}$ if and only if it satisfies both **PS** and **CC**. (In particular, every torsion-free Abelian group of size $\mathfrak{c}$ belongs to $\mathcal{C}$ under MA [16].) This result has been substantially extended in [6]:

Theorem 7. *There exists a model of* ZFC *in which the following three statements hold (simultaneously):*

(1) A group G of size at most $2^{\mathfrak{c}}$ belongs to $\mathcal{C}$ if and only if it satisfies both **PS** *and* **CC**.

(2) An Abelian group G of size at most $2^{\mathfrak{c}}$ satisfying both **PS** *and* **CC** *admits a hereditarily separable countably compact group topology without non-trivial convergent sequences.*

(3) An Abelian group G admits a countably compact separable group topology if and only if $|G| \leq 2^{\mathfrak{c}}$ and G satisfies both **PS** *and* **CC**.

This recent "jump" from $\mathfrak{c}$ to $2^{\mathfrak{c}}$ is an essential step forward with respect to the previous knowledge about countably compact group topologies on groups of size bigger than $\mathfrak{c}$. In fact, this "prior knowledge" was essentially limited to the following two facts: It is consistent with ZFC that a free Abelian group of size $2^{\mathfrak{c}}$ belongs to $\mathcal{C}$ [14] and it is also consistent with ZFC that an Abelian group of size $\aleph_\omega$ (with $\mathfrak{c} < \aleph_\omega < 2^{\mathfrak{c}}$) belongs to $\mathcal{C}$ [19].

While an algebraic description of Abelian groups admitting either a compact or a pseudocompact group topology can be carried out without any additional set-theoretic assumptions beyond ZFC, all substantial results about countably compact topologizations described above have either been obtained by means of some additional set-theoretic axioms (usually Continuum Hypothesis CH or versions of MA) or their consistency has been proved by forcing.

The lack of any ZFC results about separable countably compact topologies on Abelian groups justifies our next problem:

Problem 8. Describe in ZFC an algebraic structure of groups of size at most $2^{\mathfrak{c}}$ that belong to $\mathcal{C}$.

The following special version of this problem is also open:

Problem 9. Describe in ZFC an algebraic structure of separable countably compact Abelian groups.

The next two question provides a natural hypothesis for the solution of the above problems:

Question 10. Is it true in ZFC that an Abelian group G of size at most $2^{\mathfrak{c}}$ belongs to $\mathcal{C}$ if and only if G satisfies both **PS** and **CC**?

Question 11. Is it true in ZFC that an Abelian group G admits a separable countably compact group topology if and only if $|G| \leq 2^{\mathfrak{c}}$ and G satisfies both **PS** and **CC**?

Note that Theorem 7 provides a strong positive *consistent* answer to the last two questions.

Recall that an Abelian group G is *divisible* provided that for every $g \in G$ and each positive integer n one can find $h \in G$ such that $nh = g$. An Abelian group is *reduced* if it does not have non-zero divisible subgroups. Every Abelian group G admits a unique representation $G = D(G) \oplus R(G)$ into the maximal divisible subgroup $D(G)$ of G (the so-called *divisible part of* G) and the reduced subgroup $R(G) \cong G/D(G)$ of G (the so-called *reduced part of* G). It is well-known that an Abelian group G admits a compact group topology if and only if both its divisible part $D(G)$ and its reduced part $R(G)$ admit a compact group topology. However, there exist groups G and H that belong to $\mathcal{P}$ but neither $D(G)$ nor $R(H)$ belongs to $\mathcal{P}$ [5, Theorem 8.1 (ii)]. This was "strengthened" in [3, 2] as follows: It is consistent with ZFC that there exist groups G' and H' from the class $\mathcal{C}$ such that neither $D(G')$ nor $R(H')$ belong to $\mathcal{P}$. These results leave open the following

Problem 12. In ZFC, give an example of groups G and H from the class $\mathcal{C}$ such that:

(i) $D(G)$ does not belong to $\mathcal{C}$ (or even $\mathcal{P}$),
(ii) $R(H)$ does not belong to $\mathcal{C}$ (or even $\mathcal{P}$).

Even the following question is also open:

Question 13. Let G be a group in $\mathcal{C}$.
(i) Is it true that either $D(G)$ or $R(G)$ belongs to $\mathcal{C}$?
(ii) Must either $D(G)$ or $R(G)$ belong to $\mathcal{P}$?

We note that item (ii) of the last question is a strengthening of Question 9.8 from [5]. Even consistent results related to the last question are currently unavailable.

Assuming MA, there exist countably compact groups G, H such that $G \times H$ is not countably compact [9]. Therefore, our next question could be viewed as a "weaker form" of productivity of countable compactness in topological groups that still has a chance for a positive answer.

Question 14. If G and H belong to $\mathcal{C}$, must then their product $G \times H$ also belong to $\mathcal{C}$?

In fact, one can consider a much bolder hypothesis:

Question 15. Is $\mathcal{C}$ closed under *arbitrary* products? That is, if G_i belongs to $\mathcal{C}$ for each $i \in I$, does then $\prod_{i\in I} G_i$ belong to $\mathcal{C}$?

Our next question is motivated by a well-known theorem of Ginsburg and Saks [10] in the framework of topological spaces where $2^{\mathfrak{c}}$ can be taken for τ below:

Question 16. Is there a cardinal τ having the following property: A product $\prod_{i\in I} G_i$ belongs to $\mathcal{C}$ provided that $\prod_{j\in J} G_j$ belongs to $\mathcal{C}$ whenever $J \subseteq I$ and $|J| \le \tau$?

In particular, is $\mathfrak{c}$ or $2^{\mathfrak{c}}$ such a cardinal?

A partial positive answer to Question 15 has been given in [6]: It is consistent with ZFC that, for every family $\{G_i : i \in I\}$ of groups with $2^{|I|} \le 2^{\mathfrak{c}}$ such that G_i belongs to $\mathcal{C}$ and $|G_i| \le 2^{\mathfrak{c}}$ for each $i \in I$, the product $\prod_{i\in I} G_i$ also belongs to $\mathcal{C}$. A similar result for much smaller products and much smaller groups has been proved in [7, Theorem 5.6] under the assumption of MA. In particular, if the groups G and H in Question 14 are additionally assumed to be of size at most $2^{\mathfrak{c}}$, then the positive answer to this restricted version of Question 14 is consistent with ZFC [6].

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Title: **Strong Extension Principle**

Strong Extension Principle (sEP): Every continuous function $f\colon \omega^* \to \omega^*$ can be extended to a continuous function $g\colon \beta\omega \to \beta\omega$.

Question: *Is Strong Extension Principle consistent with* ZFC*?*

I conjecture that PFA implies the Strong Extension Principle. Shelah [1] has proved that it is consistent with ZFC that all autohomeomorphisms of ω^* are trivial. This implies that every autohomeomorphism of ω^* continuously extends to a map $f: \beta\omega \to \beta\omega$. Later Shelah-Steprans and Velickovic deduced this from PFA and weaker axiom OCA+MA, respectively. I have proved that OCA+MA imply that for every continuous $f: \omega^* \to \omega^*$ there is a clopen $U \subseteq \omega^*$ such that the restriction of f to U continuously extends to $\beta\omega$ and the image of $\omega^* \setminus U$ is nowhere dense. This is the simplest instance of the weak Extension Principle, see Chapter 4 of [2]. By ([3]), sEP is equivalent to an apparently stronger statement that for every cardinal κ every continuous $f\colon (\omega^*)^\kappa \to \omega^*$ can be extended to a continuous function $g\colon (\beta\omega)^\kappa \to \beta\omega$. K.P. Hart has observed that sEP implies that every extremally disconnected continuous image of ω^* is separable.

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Title: **Elementary sets**

Let X be a space with base $\mathcal{B}$. Sets of the form

$$E = \bigcap\{B_i : 0 \leq i < n\} \cap \bigcup\{X \setminus B_i : n \leq i < m\}$$

where each $B_i \in \mathcal{B}$ is called elementary. If we change the base $\mathcal{B}$, we (probably) change the elementary sets. Call a base $\mathcal{B}$ nice if every nonempty elementary set has nonempty interior.

Problem. *Which spaces X have nice bases?*

Clearly zero-dimensional spaces do. By the method of "distinct endpoints", $\mathbb{I}$, $\mathbb{I}^n$, and $\mathbb{I}^\omega$ do, too.

Problem. *Does every compact metric space have a nice base?*

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Title: **Metrisability of manifolds and topological games**

For a topological space X, Matveev [2] introduced the following, successively stronger, conditions:

- A space X *has property (a)* provided that for every open cover $\mathcal{U}$ of X and every dense subset $D \subset X$ there is a subset $F \subset D$ such that F is a closed and discrete subspace of X and $st(F,\mathcal{U}) = X$.
- A space X *is a-favourable* provided that for every open cover $\mathcal{U}$ of X there is a winning strategy for the second player in the following topological game: at the αth step the first player chooses a dense subspace $D_\alpha \subset X$ then the second player chooses a point $x_\alpha \in D_\alpha$; the second player wins if for some α the set $F_\alpha = \{x_\beta \;/\; \beta < \alpha\}$ is closed and discrete in X and $st(F_\alpha,\mathcal{U}) = X$.
- A space X *is strongly a-favourable* provided that for every open cover $\mathcal{U}$ of X there is a winning strategy for the first player in the following topological game: at the αth step the first player chooses a non-empty open set $O_\alpha \subset X$ then the second player chooses a point $x_\alpha \in O_\alpha$; the first player wins if for some α the set $F_\alpha = \{x_\beta \;/\; \beta < \alpha\}$ is closed and discrete in X and $st(F_\alpha,\mathcal{U}) = X$.
- A space X *has property pp* provided that every open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ consisting of non-empty sets such that for every choice function $f : \mathcal{V} \to X$ (ie $f(V) \in V$ for each $V \in \mathcal{V}$) the set $f(\mathcal{V})$ is closed and discrete in X.

Weakened versions of each property are obtained by replacing "closed and discrete" by "discrete", and are denoted by putting a w before the a or p, with obvious implications.
In [1] it is shown that every manifold is strongly wa favourable and that every manifold with property wpp is metrisable while examples are given of a manifold which does not have property (a) and of a non-metrisable manifold which is strongly a-favourable.

Problem (1)**.** *Is there a manifold which has property (a) but which is not a-favourable?*

Problem (2)**.** *Is there a manifold which is a-favourable but which is not strongly a-favourable?*

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Title: **Dowker Filters**

C.H. Dowker posed the following problem in 1952:

Problem. *Does there exist (in ZFC) a set X and a filter $\mathcal{F}$ on X such that:*

(a) *for every partition $e : X \to 2$ of X, there is an $F : X \to \mathcal{F}$ such that if $x, y \in X$ and $e(x) \neq e(y)$, then either $x \notin F(y)$ or $y \notin F(x)$;*
(b) *there is no function $F : X \to \mathcal{F}$ such that, for every $x, y \in X$, $x \neq y$ implies either $x \notin F(y)$ or $y \notin F(x)$?*

Call such a filter, if it exists, a *Dowker filter*. Dowker showed that there do not exist Dowker filters on any set of size $\leq \omega_1$. Zoli and I showed in 1991 that it is consistent for there to Dowker filters on ω_2 (among other cardinals). But it is still not known if there is a Dowker filter in ZFC.

The paper of Zoli and I on Dowker filters came after we read a paper of Mary Ellen Rudin in which she uses a certain technique for answering a topological problem related to Dowker's. This technique of Rudin is the technique that Zoli subsequently saw how to develop into a tremondously powerful method for constructing spaces, first his Q-set spaces, then his ZFC Dowker space, and so on.

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Title: **Sequentially Linearly Lindelof spaces**

A space X is *Sequentially Linearly Lindelöf* if for every set $A \subseteq X$ with $|A| = \lambda$, λ regular, uncountable and $\lambda \leq w(X)$ there is a subset $B \subseteq A$ so that $|A| = |B|$ and B *converges* to a point in X.

Problem. *Can one prove in* ZFC *alone the existence of a SLL space which is not Lindelöf?*

SLLnL spaces exist in every set-forcing extension of V = L and in every model of set theory in which the singular cardinal hypothesis fails somewhere [1].

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Title: $C_p(X)$ **and D-spaces**

For definition and discussion of D-spaces see Arhangelskii's contribution in this problem session.

Problem. *Let X be a compact space or a Lindelöf Σ-space. Must $C_p(X)$ be a D-space? a hereditarily D-space?*

Motivation: in 1980s Baturov proved that for $Y \subset C_p(X)$ where X is a Lindelöf Σ-space, $e(Y) = l(Y)$.

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Title: **Characterizing countably rectifiable curves**

Problem (1)**.** *Characterize those metrizable continua which under some compatible metric are the union of countably many sets with finite linear Hausdorff measure.*

Problem (2)**.** *Is it true that such a continuum may be embedded in R^3 so that the embedding is the union of countably many sets with finite linear Hausdorff measure with respect to the Euclidean metric?*

Reference Notes:
Eilenberg and Harrold characterized those continua which have finite linear Hausdorff measure. Fremlin showed such such continua may be embedded in R^3 to have finite linear measure.

Partial results and references may be found in “Continua with σ - finite linear measure,” Measure Theory, Oberwolfach 1990, Supplemento di Rendiconti del Circolo Matematico di Palermo, serie II, no. 38, 1992.

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Title: **A Linearly Lindelöf problem**

Problem. *Is there a linearly Lindelöf non-Lindelöf normal space of weight $\aleph_\omega$?*

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Title: **Commuting, coincidence-point-free maps on a triod**

The following question is fairly popular among continuum theorists, but I think it deserves wider recognition.

Problem. *Let T denote a simple triod—three copies of of the interval $[0, 1]$ with all the 0's identified. Does there exist a pair of continuous maps on T which commute and are coincidence-point free? I.e., do there exist continuous $f, g : T \to T$ such that $f \circ g = g \circ f$ and $f(x) \neq g(x)$ for all $x \in T$?*

If you could find such a pair of maps you would have a fantastic new continuum theory example. Let X denote the inverse limit of a sequence consisting of countably many triods, with every bonding map equal to f. Now use g to define a map $g^* : X \to X$ by:

$$g^*(\langle x_0, x_1, x_2, \dots\rangle) = \langle g(x_1), g(x_2), g(x_3), \dots\rangle.$$

Since $x_0 = f(x_1) \neq g(x_1)$, the map g^* is a fixed point free map defined on the tree-like continuum X. If you draw a diagram representing the above equation, you'll see that g^* is induced by using g

as a "diagonal" mapping from the $(i+1)$st space back to the ith space—the result is a diagram of commuting parallelograms. It is worth mentioning that you can't define such a fixed point free map with a commuting diagram of squares, since T has the fixed point property.

On the other hand, a "no" answer might be a step in proving that triod-like continuua (i.e., an inverse limit of a sequence of triods) have the fixed-point property. More importantly, it might be a step in answering the more general question "What does it mean, topologically, for a pair of continuous maps to commute?"

Now for a bit of history. David Bellamy described the first fixed point free mapping of a tree-like continuum. Shortly thereafter, Lex Oversteegen and Jim Rogers produced examples based on Bellamy's ideas which were described as inverse limits using a commuting parallelogram diagram like the one above. In these examples, the number of vertices in the factor spaces goes to infinity as $n \to \infty$ (as does as the complexity of the bonding and inducing maps). This makes it natural to ask whether there can be a "simplest" such example, with each factor space a triod and just two distinct maps. When I first heard of this question as a grad student, it was attributed to David Bellamy. However, he refuses to take credit for asking it. As far as I know, neither Lex nor Jim will take credit for asking the question either.

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Title: **Does wΔ imply $\Sigma^{\#}$?**

The title refers to two classes of "Generalized metric spaces" whose definitions are found in Gruenhage's Handbook article. The precise problem is:

Problem. *Is every regular wΔ-space $\Sigma^{\#}$?*

There is a class of generalized metrizable spaces called "β-spaces" that includes all wΔ-spaces, $\Sigma^{\#}$-spaces, and semistratifiable spaces: see Gruenhage's article again. In a way, the question is asking whether it is really redundant to be listing wΔ spaces along with the $\Sigma^{\#}$-spaces.

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Title: **Normality and point-countable bases**

Problem. *Is there a collectionwise normal space with a point-countable base that is not paracompact?*

Amazingly enough, there do not even seem to be any consistency results on this in either direction. Still more amazingly, the same is true if we add "perfectly normal" to the conditions. In this form the problem was posed as a related problem in "Classic Problems" (see [1]).

If "collectionwise" is omitted, then all our current knowledge has to do with the questions of whether every [perfectly] normal space with a point-countable base is collectionwise normal. This is because every known normal space with a point countable base has the property that if it is collectionwise normal, it is also paracompact.

This second pair of questions sits snugly between the twin issues of whether every metacompact normal Moore space is metrizable and whether every first countable normal space is collectionwise normal. On the one hand, every normal Moore space is perfectly normal, every metacompact Moore space has a point-countable base, and every collectionwise normal Moore space is metrizable. On the other hand, every space with a point countable base is first countable.

No known model or axiom distinguishes between these two issues, which are inextricably intertwined with large cardinal axioms. Yes to both is consistent if it is consistent that there is a strongly compact cardinal; but there is a metacompact nonmetrizable normal Moore space if the covering lemma holds over the Core Model. See references [2] and [3] for more on what this means.

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Title: **How strong is "hereditarily strongly cwH"?**

Problem. *Is every hereditarily strongly collectionwise Hausdorff space normal? what if the space is regular? locally compact?*

A space is said to be strongly collectionwise Hausdorff ("strongly cwH") if every closed discrete subspace expands to a discrete collection of open sets. These questions are inspired by some applications of large cardinals, including two old (1988) theorems of Zoltan Balogh, described in the abstract for my November talk; see also Reference [1]. Another application is the consistency (modulo a supercompact cardinal) of every hereditarily strongly cwH, locally compact, locally connected space being both (hereditarily) collectionwise normal and (hereditarily) countably paracompact: see Reference [3].

I am even unaware of a ZFC example of a hereditarily strongly cwH space that is not collectionwise normal. The only consistent locally compact example I know of is a normal manifold of Mary Ellen Rudin's using $\diamondsuit^+$ which is not collectionwise normal.

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Title: **Balogh's own $P = NP$ problem**

Problem. *Given a class P of normal spaces, define NP to be the class of all spaces X such that $X \times Y$ is normal for all Y in P. Is there a class P of normal spaces for which $P = NP$?*

This problem was posed by Balogh at the 1995 Spring Topology Conference in Newark, Delaware. He carefully chose the notation for obvious reasons. The more general theme of what NP is for a given P is what was behind the three Morita conjectures, two of which were solved by Balogh after he posed his $P = NP$ problem. The Morita conjectures said, in Balogh's notation, that $P = NNP$ for the classes of discrete spaces (for which NP is obviously the class of all normal spaces) and metrizable spaces, and they gave a characterization of NNP for the class of of compact metrizable spaces.

The operator N induces a Galois correspondence: P is a subclass of NNP and if P is a subclass of Q then NQ is a subclass of NP. As is usual with all Galois correspondences, $NP = NNNP$ for all P. It provides a whole general topic for problems: given a nice class P, can we find nice characterizations for NP and NNP? Of course, once we have these three, adding more N's does not introduce new classes.

Thanks to Tamano's theorem, if one begins with the class of compact spaces, then NP is the class of paracompact spaces, and it is also known that NNP is the class of sigma-locally compact paracompact spaces.

It also provides natural starting points for Balogh's $P = NP$ problem. If P is a subclass of NP, as is clearly the case where P is the class of metrizable spaces, one can try adding spaces from NP to P one at a time in the hope that as P grows and NP shrinks, they come together in a solution to Balogh's $P = NP$ problem. This simple approach will work only if all finite powers of the spaces we add to P are normal. In fact, any P which equals NP has to be closed under all finite products. So Balogh's $P = NP$ problem is equivalent to:

Is there a class P of normal spaces such that every finite product of spaces in P is normal, and such that every normal space not in P either (1) has a non-normal product with some member of P or (2) has a non-normal finite power?

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Title: **Balogh's New Classic Problem**

Preface

This note has been adapted from Zoltan Balogh's contribution to *New Classic Problems* [2] and from the introduction in Balogh's article *On two problems concerning Baire sets in normal spaces* [1]. This note will appear in the collection *Problems from Topology Proceedings* [7].

A Problem of Katětov

Given a topological space X, let Borel(X) and Baire(X) denote the σ-algebras generated by the families closed$(X) = \{F : F \text{ closed in } X\}$ and zero$(X) = \{F : F \text{ is a zero set in } X\}$, respectively. The following question is due, without the phrase "in ZFC", to M. Katětov [6].

Problem (M. Katětov [6])**.** *Is there, in* ZFC*, a normal T_1 space X such that* Borel(X) = Baire(X) *but X is not perfectly normal (i.e.,* closed$(X) \neq$ zero(X)*)?*

What if X is also locally compact? first countable? hereditarily normal?

Notes. There are several consistency examples given by Z. Balogh in [1]. CH implies that there is a locally compact locally countable X satisfying the conditions of the problem. The existence of a first countable, hereditarily paracompact X is consistent, too.

However, as summarized by the following theorem, a space giving a positive answer to the question cannot satisfy certain properties.

Theorem. *Let X be a normal T_1 space, and let A be a closed Baire subset of X. Then A is a zero set in X if one of the following conditions hold.*

- *X is compact* (P. R. Halmos [4])*.*
- *X is paracompact and locally compact* (W. W. Comfort [3])*.*
- *X is submetacompact and locally compact* (D. Burke)*.*
- *X is Lindelöf and Čech-complete* (W. W. Comfort [3])*.*
- *X is a subparacompact $P(\omega)$-space* (R. W. Hansell [5])*.*

In [1], Balogh gave a counterexample to the following related question of K. A. Ross and K. Stromberg. The construction makes use of the technique of E. K. van Douwen and H. H. Wicke [9] and W. Weiss [10]

Problem (K. A. Ross and K. Stromberg [8])**.** *If X is a a normal locally compact Hausdorff space and A is a closed Baire set in X, is A a zero set?*

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Title: **Para-Lindelöf collectionwise normal spaces**

Problem. *Are para-Lindelöf collectionwise normal spaces paracompact?*

This was first asked by W. Fleissner and G. M. Reed [2]. It is Problem 109 from S. Watson's list [4]

Problem. *Are meta-Lindelöf, collectionwise normal space paracompact?*

This is R. Hodel's question [3] and also Problem 110 from Watson's list.

Zoltan Balogh [1] constructed a hereditarily collectionwise normal, hereditarily meta-Lindelöf, hereditarily realcompact Dowker space.

Balogh listed some open questions about meta-Lindelöf and para-Lindelöf Dowker spaces at the end of his article [1].

(1) Is there a para-Lindelöf, collectionwise normal Dowker space?
(2) Is there a para-Lindelöf Dowker space?
(3) Is there a meta-Lindelöf, collectionwise normal and first countable Dowker space?
(4) (D. Burke) Is there a meta-Lindelöf, collectionwise normal and countably paracompact space which is not paracompact?
(5) Is there a first countable Dowker space in ZFC?

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Title: **Embedding free topological groups into unitary groups**

Problem. *Let $F(X)$ denote the (Markov or Graev) free topological group on a compact metric space X. Is it isomorphic with a topological subgroup of the group $U(\ell_2)$ of all unitary operators of the separable Hilbert space, equipped with the strong operator topology (that is, the topology induced from the usual Tychonoff power $(\ell_2)^{\ell_2}$)? What about the free topological group on the convergent sequence with the limit? In fact, the author of the present problem does not even know if the free topological group $F(X)$ on an arbitrary Tychonoff space X embeds into $U(\ell_2)$ as a topological subgroup.*

Embeddability of a topological group G into $U(\ell_2)$ with the strong topology is equivalent to the fact that continuous positive definite functions on G separate identity from closed subsets not containing it, cf. for instance the paper by Megrelishvili [1] and references therein.

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E-print available from: http://www.cs.biu.ac.il/~megereli/nunit.ps

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Title: **Bing points of countable-dimensional continua**

A space is countable-dimensional if it is the union of countably many finite-dimensional subsets. By a continuum we mean a metric compact connected space, and dim stands for the dimension. A continuum X is hereditarily indecomposable, if for any two intersecting subcontinua K and L of X, either $K \subset L$ or $L \subset K$. If X is an infinite-dimensional continuum then by $B_\infty(X)$ we denote the set of Bing points of X, i.e. the set of points x such that every non-trivial subcontinuum of X containing x is infinite-dimensional.

Problem. *Does there exist for every integer $n \geq 2$ a countable-dimensional (if possible, hereditarily indecomposable) continuum X with $\dim B_\infty(X) \geq n$?*

For $n \in \{-1, 0, 1\}$, examples of countable-dimensional infinite-dimensional hereditarily indecomposable continua X with $\dim B_\infty(X) = n$ were constructed in [PR].

The positive answer to this problem would yield the negative answer to the following

Problem (R.Engelking and E.Pol [EP]). . *Can every compact metrizable countable-dimensional space be mapped onto a finite-dimensional space by a mapping with finite-dimensional fibers?*

Indeed, assume that X_n is a countable-dimensional continuum such that $\dim B_\infty(X_n) \geq n$ and let $f : X_n \to Y$ be a mapping onto a continuum Y with finite-dimensional fibers. Then using a reasoning involving the monotone-light factorization of f, one checks that $\dim Y \geq n$. In effect, the one point compactification of the free union $\bigoplus_{n=2}^{\infty} X_n$ of the spaces X_n cannot be mapped onto a finite-dimensional space by a mapping with finite-dimensional fibers.

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Title: **Base-base, base-cover and base-family paracompactness**

Problem (1). *Is every subspace of the Sorgenfrey line base-base paracompact?*

A space is base-base paracompact (defined by John Porter in [9] under a different name) if it has an open base such that every subfamily which is still a base has a locally finite subcover. Although he was mainly interested in another property (base paracompactness, see also [10]), he observed in [9] that every base-base paracompact space is a D-space, and that all metric spaces, all non-archimedean spaces, and the Sorgenfrey line are base-base paracompact. (The above problem was implicit in [9] and was discussed by John Porter and other topologists in Auburn before he left for Murray State University. It is still unknown if every paracompact space is base-base paracompact [9].) In [11] we called a space base-cover paracompact (a stronger property) if it has an open base every subcover of which has a locally finite subcover, and proved that a subspace of the Sorgenfrey line is F_σ iff it is base-cover paracompact. It is known (including work of Zoltan Balogh) that Lusin subspaces of the Sorgenfrey line and, under MA, subspaces of cardinality less than continuum are Hurewicz [1], [4], [5], and therefore totally paracompact [3] (i.e. every base has a locally finite subcover).

The next problem was formulated by Gary Gruenhage after the author stated that base-family paracompactness is a generalized metrization property and solved some related problems. A space is base-family paracompact (defined in [11] under a different name) if it has an open base every subfamily of which contains a subfamily with the same union such that the latter subfamily is locally finite at each point of that union. Every proto-metrizable space (in particular every non-archimedean space) is base-family paracompact [11].

Problem (2)**.** *Is every compact, perfectly normal, base-family paracompact space metrizable?*

There are consistent examples of perfectly normal, non-archimedean and non-metrizable spaces, e.g. the branch space of a Souslin tree, see [8], section 7 of [6], [12], [2]. Compact proto-metrizable spaces are metrizable [8]. Perfectly normal, non-metrizable compacta are discussed in [7]. The Alexandroff double arrow is not base-family paracompact; the one-point compactification of an uncountable discrete space is base-family paracompact but not perfectly normal [11].

Problem (3)**.** *Is every base-family paracompact space monotonically normal?*

There are more questions related to these new classes of spaces in [9] and [11] (see also [10] and two forthcoming papers of the author about base-cover and base-family paracompactness).

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Title: **Some old favorites**

Problem (1)**.** *If a normal T_2 space is the union of countably many open metrizable spaces, is it metrizable?*

I have shown the answer yes for card $< \mathfrak{c}$ under various set-theoretic assumptions. I also have a T_3 space example which is not a Moore space in ZFC.

Problem (2)**.** *Does there exist a* ZFC *example of a T_3 perfect space which does not have a σ-discrete dense subset; in particular does there exist a first countable one?*

Call a space *star-compact* if each open cover $\mathcal{U}$ has a finite subset H such that the set V of all elements in $\mathcal{U}$ which have non-empty intersection with an element of H is also a cover.

Problem (3)**.**

a) *Is each star-compact Moore space countably compact?*
b) *Does there exist a first countable, star-compact T_3 space that is not countably compact?*

The answer to both a) and b) is yes under CH.